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Validity of the one-body current for the calculation of form factors in the point form of relativistic quantum mechanics

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Abstract. Form factors are calculated in the point form of relativistic quantum mechanics for the lowest energy states of a system made of two scalar particles interacting via the exchange of a massless boson. They are compared to the exact results obtained by using solutions of the Bethe-Salpeter equation which are well known in this case (Wick-Cutkosky model). The relevance of the comparison is examined by considering other relativistic quantum-mechanics approaches where results are known or have been obtained recently. Deficiencies of the point-form approach together with the single-particle current are emphasized. They point to quite sizeable contributions of two-body currents. These ones are required to fulfil current conservation in any case and to reproduce the high momentum transfer behaviour expected from the Born amplitude.

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1 Introduction

Calculations of form factors often retain in a first approximation the single-particle current (impulse approximation). In most approaches, this contribution has to be completed by at least two-body currents, especially to fulfil current conservation. This also holds for relativistic approaches where a Lorentz-covariant calculation of the single-particle current does not necessarily imply a proper account of the whole physical process, which, among other things, can contain contributions that may be considered as relativistic ones in a different formalism. For each relativistic approach, it is therefore important to test the degree of validity by comparing its predictions to a case where an "exact" calculation can be performed, as far as the full relativistic covariance properties are concerned. This especially includes the independence of the space-time surface chosen to describe the dynamics in the various forms of relativistic quantum mechanics, for instance [1]. To make the comparison relevant and instructive, this calculation should also incorporate minimal ingredients related to the description of the physical process, like boson-exchange–based interactions.

It is generally considered that the Bethe-Salpeter equation with a one-boson exchange kernel provides such a reference calculation. This approach has been used in its full complexity by Zuilhof and Tjon, for instance [2], for the deuteron case. It currently represents a necessary step in the description of a two-body system, often made with approximations however. In the simplest case, the Bethe-Salpeter equation can only involve a one-boson exchange kernel with spinless particles and constant form factors. The single-particle current has then a unique expression and is conserved. This defines a minimal interaction model whose properties can be reproduced in an instant-form formalism for instance, by considering the sum of all time-ordered diagrams corresponding to what the Bethe-Salpeter equation allows one to sum up. It may not be relevant as far as the full field theory underlying the model is concerned. Many corrections should be considered like renormalized vertex form factors, crossed diagrams, off-shell effects, etc. They are essential for a comparison to an experiment. However, all of them will similarly affect a relativistic quantum-mechanics approach. For a test like the one we have in mind, aiming to see whether minimal physics ingredients together with relativity are properly accounted for, they cancel out. In some sense, this "exact" calculation will play the role of an experiment, whose underlying physics can be tuned at will, however. Thus, to make the test, one has only to worry about the derivation of the interaction and the associated currents to be used in the particular form of relativistic quantum mechanics, due to the effective character of the related degrees of freedom that such an approach implies. Concerning the interaction,

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some work going beyond a phenomenological parametrisation has been done recently for the instant-form case [3]. Concerning the currents, which could contain for instance contributions due to removing off-shell effects in the original interaction, the task is much more complicated. For the time being, only the comparison of predictions with the "exact" calculation can tell us about their importance.

A particular case of interest for our purpose is the Wick-Cutkosky model [4,5], where the Bethe-Salpeter equation [6] is solved with an interaction kernel resulting from the exchange of a scalar zero-mass boson between two distinguishable scalar particles (one-boson ladder approximation). Solutions can be obtained relatively easily due to an extra hidden symmetry. Using the expressions of the Bethe-Salpeter amplitudes, calculations of form factors can be performed for the lowest bound states that we intent to consider here. Contrary to other approaches, there is no need to add two-body currents. The contribution of the single-particle current is sufficient in the present model to ensure current conservation, which can be checked with the expressions of the matrix elements of the current[1]. This model was used by Karmanov and Smirnov as a test of the description of form factors in the light-front approach for systems composed of scalar particles with small binding energy [8]. They used the non-relativistic expression, $-\alpha^2/4n^2$, for the binding energy, which differs from the exact one, but this does not seem to affect their conclusion. Interestingly enough, the comparison of the "exact" calculation with the non-relativistic one does not show much difference up to $Q^2 \simeq 100\, m^2$, where m is the constituent mass. Beyond, relativistic corrections with a log character slowly begin to show up.

We propose to make a similar test for the point form of relativistic quantum mechanics, which is one of the forms proposed by Dirac, beside the instant and the front forms [1]. This form, which is much less known than the other two, was first advocated by Sokolov [9] and applied later to the calculation of form factors by Lev [10] and Klink [11]. It has been recently used for a calculation of form factors of the deuteron [12] and the nucleon [13]. In both cases, the form factors decrease faster with Q^2 than the non-relativistic ones. In the first case, the discrepancy with experiment tends to increase while, in the other one, it almost vanishes in the lower Q^2 range considered by the authors. For the highest Q^2, one can guess from their results an under-prediction which tends to increase with the momentum transfer. We will not extend in this paper on the questions that are raised by these observations. We will only present a few results which, by themselves, are quite instructive. They are obtained using an approach identical to the one employed in refs. [12,13], except for spin complications that are absent here. A more complete analysis of the results will be presented elsewhere [14].

In the present study, we calculate the ground-state ($l = 0$) form factor as well as a transition form factor to the first radial excited state. Different couplings are considered, corresponding to states weakly, moderately and strongly bound ($\alpha = 1,\ 3$ and 2π in terms of the QED coupling). The last value implies a total zero mass of the system, which is an extreme (unphysical) case, nevertheless interesting to look at, too. Momentum transfers up to 100 times the constituent mass squared are considered. These different cases will provide a sample of results which are significant enough to test the validity of the single-current approximation in the point-form approach and give insight into the results presented in refs. [12,13]. Some attention will be given in particular to the limits $M \to 0$ and $Q^2 \to \infty$, which a relativistic approach should be able to deal with. As often done, we will compare the results of this point-form approach with "non-relativistic" ones. These last ones are essentially characterized by the Galilean boost that allows one to relate systems in different frames, while the description of the system itself may already involve some relativity. We do not expect them to do well but they provide benchmarks that a relativistic approach should improve upon.

2 Expression of the single-particle matrix element in different approaches

The contribution which we are interested in is shown in fig. 1. The general expression of the corresponding matrix element between two states with $l = 0$, possibly different, is given by

$$\sqrt{2E_f\, 2E_i}\ \langle f|J^\mu|i\rangle = F_1(q^2)\,(P_f^\mu + P_i^\mu) + F_2(q^2)\,q^\mu, \quad (1)$$

where $q^\mu = P_f^\mu - P_i^\mu$. Current conservation imposes constraints on the form factors $F_1(q^2)$ and $F_2(q^2)$. For an elastic process, $F_2(q^2)$ has to vanish but this result automatically stems from symmetry arguments alone. It does not imply that current conservation holds at the operator level, as it should. For an inelastic process, the following relationship has to be fulfilled:

$$F_1(q^2)\,(M_f^2 - M_i^2) + F_2(q^2)\,q^2 = 0. \quad (2)$$

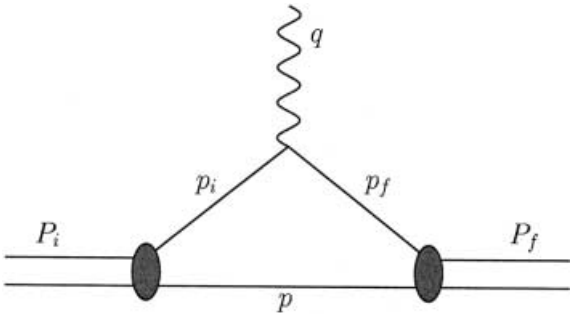

Fig. 1. Representation of a virtual-photon absorption on a two-body system with the kinematical definitions.

[1] This result together with some of the expressions used in the present paper for the Wick-Cutkosky model are part of a work in preparation. A preliminary presentation was made in [7].

2.1 Form factors using the Bethe-Salpeter amplitudes

For the model under consideration here, the general (and exact) expression of the matrix element of the current, which reduces to a single-particle one in this case, can be written in terms of the Bethe-Salpeter amplitudes, $\chi_{_P}(p)$,

$$\sqrt{2E_f\,2E_i}\ \langle f|J^\mu|i\rangle = i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\ \bar\chi_{_{P_f}}\left(\frac{1}{2}P_f-p\right)\times\left(P_f^\mu+P_i^\mu-2\,p^\mu\right)\ (p^2-m^2)\ \chi_{_{P_i}}\left(\frac{1}{2}P_i-p\right). \quad (3)$$

For the Wick-Cutkosky model, the Bethe-Salpeter amplitudes take the form of a relatively simple integral representation, *i.e.* for the lowest energy state with a given orbital angular momentum l:

$$\chi_{_P}(p)=\int_{-1}^{1}\mathrm{d}z\ \frac{g_n(z)\ Y_l^m(\hat p)\ |\mathbf{p}\,|^l}{(m^2-\frac{1}{4}P^2-p^2-z\,P\cdot p-i\epsilon)^{n+2}}, \quad (4)$$

with $n=l+1$. In this expression, $g_n(z)$ is the solution of a second-order differential equation [4,5], that can be solved easily.

2.2 Form factors in the non-relativistic limit

In contrast with a full relativistic calculation, a non-relativistic one can be easily performed. Using wave functions that are solutions of a Schrödinger equation, general expressions for both elastic and inelastic form factors can be obtained. In the Breit frame, they read

$$F_1(q^2)=\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\ \phi_f\left(\mathbf{p}-\frac{1}{4}\mathbf{q}\right)\ \phi_i\left(\mathbf{p}+\frac{1}{4}\mathbf{q}\right),$$
$$F_2(q^2)\,\frac{\mathbf{q}}{4m}=-\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\times\ \phi_f\left(\mathbf{p}-\frac{1}{4}\mathbf{q}\right)\ \frac{\mathbf{p}}{m}\ \phi_i\left(\mathbf{p}+\frac{1}{4}\mathbf{q}\right). \quad (5)$$

It can be checked that the above form factors verify the current conservation condition, eq. (2), provided the interaction is local. The second form factor vanishes in the elastic case.

Actually, expressions given by eqs. (5) can be used with wave functions issued from an equation going beyond the non-relativistic Schrödinger one, like the one we will use in practice in the following, which involves some relativity (see eq. (7)). The above form factors are then obtained by relying on a Galilean boost. When referring in the following to a "non-relativistic" calculation of form factors, we will have mainly in mind this boost transformation though the wave function has been obtained from an equation with a relativistic kinetic energy. The conserved current associated to this equation together with a local interaction does not affect the expressions of the form factors, eqs. (5).

2.3 Form factors in the point-form approach

It has been shown [11] that a calculation of form factors in the point-form approach could be performed relatively easily by using standard wave functions obtained from a mass operator of the form

$$M=M_{\rm free}+M_{\rm int}. \quad (6)$$

This includes a large class of wave functions, since the sum of the kinetic and potential energies appearing in the standard Schrödinger equation can be identified with the operator M^2 (up to a factor). This holds for a two-body system and provided the binding energy, E, is appropriately redefined. Therefore, wave functions entering the non-relativistic expressions of eq. (5) could be used in principle. Actually, we will use wave functions obtained from a mass operator of the type

$$M^2=4\,(m^2+\mathbf{p}^{\,2})+4m\,V, \quad (7)$$

which has a form similar to a Schrödinger equation, but does not require the above energy correction.

In calculating the matrix element of the current, we follow the approach used in ref. [12]. The single-particle current is taken as the one expected from the field theory underlying the "exact" calculation, *i.e.* the free-particle one. This choice is dictated by the requirement to recover the small coupling limit but beyond this limit there is no reason this should provide the best description of the one-body current. Possible corrections, involving off-shell effects, could be absorbed in two-body currents. Other two-body currents could account for the effectiveness of the degrees of freedom that relying on (relativistic) quantum mechanics implies, for current conservation or for reproducing the Born amplitude. Further ones, which ensure the equivalence of different representations of the complete interaction obtained by a unitary transformation, do not need to be considered, provided that an appropriate choice of the effective two-body interaction is made. All of them cancel at $q^\mu=0$. On the other hand, the normalization of the current to the charge and the related orthogonality condition should be recovered. These various requirements fix the expression of the single-particle matrix element. Contrary to ref. [12], where appropriate boosts have to be performed, we provide an expression whose Lorentz covariance is explicit:

$$\begin{aligned}&\sqrt{2E_f\,2E_i}\ \langle f|J^\mu|i\rangle=\\&\sqrt{2M_f\,2M_i}\ \frac{1}{(2\pi)^3}\int\mathrm{d}^4p\,\mathrm{d}^4p_f\,\mathrm{d}^4p_i\,\mathrm{d}\eta_f\,\mathrm{d}\eta_i\\&\quad\times\delta(p^2-m^2)\ \delta(p_f^2-m^2)\ \delta(p_i^2-m^2)\\&\quad\times\theta(\lambda_f\cdot p_f)\ \theta(\lambda_f\cdot p)\ \theta(\lambda_i\cdot p)\ \theta(\lambda_i\cdot p_i)\\&\quad\times\delta^4(p_f+p-\lambda_f\eta_f)\ \delta^4(p_i+p-\lambda_i\eta_i)\\&\quad\times\phi_f\left(\left(\frac{p_f-p}{2}\right)^2\right)\ \phi_i\left(\left(\frac{p_i-p}{2}\right)^2\right)\\&\quad\times\sqrt{(p_f+p)^2\,(p_i+p)^2}\quad(p_f^\mu+p_i^\mu).\end{aligned} \quad (8)$$

In this expression, all quantities are Lorentz-invariant ones (except obviously the current, which behaves as a four-vector). The auxiliary variables, η_i and η_f, have been introduced to make the covariance manifest[2]. When these variables are integrated over, they give rise to three-dimensional $\delta(...)$-functions,

$$\delta\Big(\mathbf{p}_i+\mathbf{p}-\frac{\boldsymbol{\lambda}_i}{\lambda_i^0}\left(p_i^0+p^0\right)\Big) \text{ and } \delta\Big(\mathbf{p}_f+\mathbf{p}-\frac{\boldsymbol{\lambda}_f}{\lambda_f^0}(p_f^0+p^0)\Big),$$

that are essential relations in the point-form approach. The λ_i^μ and λ_f^μ four-vectors are unit vectors proportional to the four-momenta of the total system in the initial and final states, $\lambda_i^\mu = P_i^\mu/M_i$ and $\lambda_f^\mu = P_f^\mu/M_f$. They can be expressed in terms of the corresponding velocities[3], $\lambda^0 = (\sqrt{1-v^2}\,)^{-1}$ and $\boldsymbol{\lambda} = \mathbf{v}\,(\sqrt{1-v^2}\,)^{-1}$ ($c=1$). In the c.m., it can be checked that the wave function $\phi(...)$ only depends on the relative three-momentum of the two particles. On the other hand, it can also be verified, by direct integration or after performing a change of variable, that the current of the system under consideration is given by $(\langle J^0\rangle\,,\,\langle\mathbf{J}\,\rangle) = (1\,,\,\mathbf{v}\,)$, in agreement with the standard normalisation of the wave function

$$\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\phi^2(\mathbf{p}\,) = 1.$$

Expressions of the form factors, $F_1(q^2)$ and $F_2(q^2)$, can be obtained in any frame. With an appropriate change of variables, they can always be expressed in the forms they take in the Breit frame (here defined by $\mathbf{v} = \mathbf{v}_f = -\mathbf{v}_i$), where they may be simpler. Using auxiliary quantities, $\tilde{F}_1(q^2)$ and $\tilde{F}_2(q^2)$, they read

$$\begin{aligned} F_1(q^2)\,\sqrt{2M_f\,2M_i} &= \\ &\tilde{F}_1(q^2)\,(M_f+M_i) - \tilde{F}_2(q^2)\,(M_f-M_i), \\ F_2(q^2)\,\sqrt{2M_f\,2M_i} &= \\ &-\tilde{F}_1(q^2)\,(M_f-M_i) + \tilde{F}_2(q^2)\,(M_f+M_i), \end{aligned} \tag{9}$$

with

$$\tilde{F}_1(q^2) = \frac{1+v^2}{\sqrt{1-v^2}}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\phi_f(\mathbf{p}_{tf})\,\phi_i(\mathbf{p}_{ti}),$$

$$\tilde{F}_2(q^2)\,\mathbf{v} = -\frac{1+v^2}{\sqrt{1-v^2}}\int \frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\,\phi_f(\mathbf{p}_{tf})\frac{\mathbf{p}}{e_p}\,\phi_i(\mathbf{p}_{ti}). \tag{10}$$

In the above equations, the velocity $\mathbf{v}$, defined in the Breit frame, is related to the momentum transfer by the relation

$$v^2 = \frac{Q^2+(M_f-M_i)^2}{Q^2+(M_f+M_i)^2}. \tag{11}$$

[2] We keep here a definition of covariance that is often used in the literature. Nevertheless, despite its appearance, a careful examination would evidence an implicit dependence on the reference frame used to calculate the original wave function.

[3] The quantity, v, here refers to the usual velocity three-vector, $\mathbf{v}$, a notation that differs from the one employed in ref. [11], where it represents a four-vector which corresponds to our four-vector λ^μ.

The (Lorentz-) transformed momenta are defined as

$$(p^x,p^y,p^z)_{tf} = (p^x,p^y,(p^z-v\,e_p)/\sqrt{1-v^2}\,),$$

$$(p^x,p^y,p^z)_{ti} = (p^x,p^y,(p^z+v\,e_p)/\sqrt{1-v^2}\,),$$

together with $e_p = \sqrt{m^2+\mathbf{p}^{\,2}}$.

2.4 Wave functions and analytical results

For the wave functions of the ground and first excited states, $\phi(\mathbf{p}\,)$ and $\phi^*(\mathbf{p}\,)$, we use solutions obtained from eq. (7) with a Coulomb-like potential, $V = -\frac{\alpha_{\rm eff}}{r}$. They therefore fulfil the equation

$$\left[4(m^2+\mathbf{p}^{\,2})-M^2\right]\,\phi(r) = 4m\,\frac{\alpha_{\rm eff}}{r}\,\phi(r). \tag{12}$$

In momentum space, they read

$$\begin{aligned} \phi(\mathbf{p}\,) &= \sqrt{4\,\pi}\frac{4\,\kappa^{5/2}}{(\kappa^2+\mathbf{p}^{\,2})^2}, \\ \phi^*(\mathbf{p}\,) &= \sqrt{4\pi}\frac{8\,\kappa^{*5/2}}{(\kappa^{*2}+\mathbf{p}^{\,2})^3}(\mathbf{p}^{\,2}-\kappa^{*2}), \end{aligned} \tag{13}$$

where $\kappa^2 = m^2 - \frac{1}{4}M^2$, the total mass M being that one obtained from the Bethe-Salpeter equation for the ground state.

It has been shown [15,3] that the spectrum of the normal states for the Wick-Cutkosky model could be reproduced with a three-dimensional equation and a theoretically motivated effective interaction of the Coulomb-type like the one used here. In particular, a relation between the effective coupling, $\alpha_{\rm eff}$, appearing in eq. (12) and the one employed in the Bethe-Salpeter equation, α, has been derived for any value it can take [3]. This relationship was checked in detail in the non-relativistic limit and its validity was found to extend much beyond. When employed in eq. (12), it provides a spectrum of normal states for the Wick-Cutkosky model that does not differ by more than a factor 2 in the limit $\alpha_{\rm eff} \to \infty$. Relatively minor adjustments of the effective coupling are thus required to reproduce the exact spectrum.

However, it is known that once an interaction model reproduces experimental data such as binding energies or phase shifts, a continuum of models having the same properties can be generated by applying a unitary transformation. These ones differ from the original one by their non-locality which has typically an off-shell character. In a phenomenological approach or in the absence of an underlying theory, the resulting uncertainty should be considered. In the present case, we have such a theory and while we do not expect the bare interaction to be fully appropriate, we at least expect it to be recovered in the small coupling limit. This allows one to discard a continuum of phase equivalent interaction models which, beside their arbitrariness, would have the wrong off-shell or non-local behavior. This is not sufficient to totally eliminate any uncertainty in principle. In deriving the effective interaction

to be used in eqs. (6) or (7), one has to consider the series of the time-ordered diagrams that the Bethe-Salpeter equation allows one to sum up and get rid of their dependence on the total energy of the system, which characterizes the coupling of channels with a different number of particles. The procedure used in refs. [3,15] reminds one of the Foldy-Wouthuysen transformation and, like this one, provides a unique answer. One can imagine to carry out the elimination of the energy dependence (the coupling of large and small components for the Foldy-Wouthuysen transformation) by mixing terms of different orders. In this way, one generates a family of effective interactions which should be equivalent up to a unitary transformation. For the present problem however, where only one kind of exchanged boson with a unique coupling is considered, there is no physical support to proceed that way, which would appear as completely arbitrary and unnatural. Assuming one nevertheless does so, one should also take into account that the interaction with the external probe will be affected by the same unitary transformation with the result that predictions for observables are not changed. Considering other interaction models, equivalent up to a unitary transformation, is therefore an unnecessary complication for our purpose. Results obtained with the effective interaction of eq. (12) will thus represent those of a family of "phase-equivalent" models.

While we believe that the non-uniqueness of the effective interaction is not a real source of concern, some approximations in its derivation should be considered. At the lowest order where it was performed, the effective interaction, which mainly accounts for retardation effects related to the underlying field theory model, keeps a local character. Some non-locality is expected, however. It is difficult to estimate its effect, but taking into account that it would *a priori* break the degeneracy pattern of the Coulomb potential evidenced by the Wick-Cutkosky model and that the effective interaction in eq. (12) already provides a good description of the spectrum, not much room is left for such a contribution when eq. (12) is used. The situation is somewhat different for eq. (6). It is easy to convince oneself that, to get the same spectrum as eq. (12), the interaction part of the mass operator, $M_{\rm int}$, has to be non-local in any case, while wave functions are unchanged.

A complementary and instructive information on the derivation of effective interactions is given by the Coulomb potential itself. This one supposes to consider the ladder diagrams generated by solving the Bethe-Salpeter equation but also an infinite set of crossed diagrams that essentially have the effect to cancel the renormalization that the effective coupling in eq. (12) accounts for. Apart from these extra contributions, the effective interaction can be obtained as described above for the case of a scalar boson. There is no indication that the dynamical effects under discussion here provide a large contribution. The Coulomb part of the interaction due to a photon exchange appearing in a Dirac or Klein-Gordon equation is generally described by a $1/r$ term.

It follows from the above discussion that wave functions, eqs. (13), should be a good zeroth-order approximation for our study, including the extreme case $M^2 = 0$. Accordingly, we assume $\kappa^{*2} = 0.25\,\kappa^2 = 0.25\,(m\alpha_{\rm eff}/2)^2$, which differs from the Bethe-Salpeter result, $\kappa^{*2} = m^2 - \frac{1}{4}M^{*2}$, by a few percent. With these wave functions, it turns out that the form factors can be calculated analytically, allowing one to get insight into the main features evidenced by the results. We nevertheless checked that these properties, for the most striking ones, are insensitive to this choice. For that, we used a different (numerical) wave function [14], more in the spirit of the point-form approach advocated in ref. [12], *i.e.* obtained directly from the linear mass operator, eq. (6). The effective interaction employed in this calculation is closer to what field theory suggests,

$$V_{\rm int}(\mathbf{p},\mathbf{p}\,') = -\frac{m}{e_p}\,\frac{g^2}{(\mathbf{p}-\mathbf{p}\,')^2}\,\frac{m}{e_{p'}}\,,$$

and its strength is fitted to reproduce the ground-state energy of the Wick-Cutkosky model. While it involves some non-locality, allowing one to get insight into the corresponding effect discussed above, it does not do as well as eq. (12) for the spectrocopy of the excited states.

In the non-relativistic case, the elastic and inelastic form factors are, respectively, given by

$$\begin{aligned} F_1^{\rm el}(q^2) &= \frac{\kappa^4}{(\kappa^2+Q^2/16)^2}, \qquad F_2^{\rm el}(q^2) = 0, \\ F_1^{\rm inel}(q^2) &= \sqrt{2}\,\frac{64\,\kappa^4\,Q^2}{(9\kappa^2+Q^2)^3}, \\ F_2^{\rm inel}(q^2) &= \sqrt{2}\,\frac{192\,\kappa^6}{(9\kappa^2+Q^2)^3}. \end{aligned} \tag{14}$$

Taking into account that $M_f^2 - M_i^2 = 3\,\kappa^2$, one can verify that the condition of current conservation, eq. (2), is fulfilled.

In the point-form approach, the elastic form factor, written in a way that resembles the non-relativistic one, reads

$$\begin{aligned} F_1(q^2=-Q^2) &= \frac{\kappa^4\,\left(1+2\,\frac{Q^2}{4M^2}\right)}{\left(\kappa^2+\frac{Q^2}{16\left(1+\frac{Q^2}{4M^2}\right)}\right)^2\,\left(1+\frac{Q^2}{4M^2}\right)^4}, \\ F_2(q^2=-Q^2) &= 0. \end{aligned} \tag{15}$$

Interestingly, the factor $1/(1+\frac{Q^2}{4M^2})$ which multiplies the quantity $Q^2/16$ in the denominator of $F_1(q^2)$ is the same as the one sometimes introduced by hand in order to account for the Lorentz-contraction effect (see discussion in ref. [16]). Contradicting asymptotic results (in QCD for instance), the range of validity of this recipe is limited to small Q^2. Curiously, part of the extra factors also look like a Lorentz-contraction effect.

As for the inelastic form factors for a transition from the ground to the first radially excited state, they are most

easily expressed in terms of the quantities, $\tilde{F}_1(q^2)$ and $\tilde{F}_2(q^2)$, which can also be calculated analytically:

$$\tilde{F}_1(q^2) = \sqrt{2}\,(1+v^2)\,(1-v^2)^3 \times \frac{64\,\kappa^4\,v^2\,(16\,m^2 - 4\kappa^2(1-v^2))}{(9\kappa^2 + v^2\,(16\,m^2 - 10\kappa^2) + v^4\kappa^2)^3},$$

$$\tilde{F}_2(q^2) = \sqrt{2}\,(1+v^2)\,(1-v^2)^4 \times \frac{64\,(3+v^2)\,\kappa^6}{(9\kappa^2 + v^2\,(16\,m^2 - 10\kappa^2) + v^4\kappa^2)^3}, \qquad (16)$$

where v^2 is defined by eq. (11). These expressions generalize those for the non-relativistic case, eq. (14) (we recall that in this limit $Q^2 = 16\,v^2\,m^2$). In contrast however, they do not allow us to fulfil current conservation, eq. (2).

3 Results

In table 1, results are presented for elastic form factors corresponding to small ($\alpha = 1$), moderate ($\alpha = 3$) and strong binding ($\alpha = 2\pi$). In the last case, the elastic form factors in the point form vanish at finite Q^2 and results are actually those obtained when approaching the limit $\alpha \to 2\pi$, with $E = 2\,m - M = 1.90\,m$ and $E = 1.95\,m$. For these energy values, the "non-relativistic" results, to which the previous ones may be compared, are essentially the same as for $E = 2.0\,m$, given in the table.

One immediately notices that the "non-relativistic" results are very close to the "exact" ones for all cases, including the extreme case where the total mass of the system is zero. This agreement extends up to $Q^2 = 100\,m^2$ with an error of 20% for $\alpha = 1$ and 50% for $\alpha = 2\,\pi$. The discrepancy at small momentum transfers is typically of the order of $Q^2/16m^2$. Most probably, it can be traced back to the wave function used in the "non-relativistic" calculation or to the electromagnetic single-particle current. These ingredients do not fully account for corrections due to factors m/e (in the potential for instance) or for the field theory character of the Wick-Cutkosky model. A major lesson of these results is that relativistic effects are not necessarily important and that most probably, for any non-relativistic calculation of form factors of systems composed of spinless particles, there exists a covariant calculation (the exact one in the present case) which gives close results over a large range of momentum transfers. However, this does not mean that this calculation involves all relativistic effects and is physically relevant, especially with respect to current conservation. An example is provided by the deuteron electrodisintegration near threshold in the light-front approach. Two covariant calculations of the transition form factors have been shown to be very close to the non-relativistic ones up to $Q^2 = 10$ (GeV/c)^2 [17], completely missing the contribution due to the pair term whose relativistic character is well known. In the non-relativistic approach, this contribution, which is required to fulfil current conservation, is essential to account for experiment in the low momentum transfer range. How this contribution appears in the covariant light-front formalism was shown later on [18].

Table 1. Elastic form factor, $F_1(q^2)$, for the ground state. Units for E and κ are the constituent mass, m. Results are presented for different couplings and, for each of them, for different approaches: Bethe-Salpeter equation (B.S.), "non-relativistic" calculation (N.R.) and point-form approach (P.F.). The value of α referred to in the table corresponds to the Bethe-Salpeter equation, while the theoretical coupling for the "non-relativistic" model, $\alpha_{\rm eff}$, is slightly modified to reproduce the exact binding energy for the ground state.

Q^2/m^2	0.01	0.1	1.0	10.0	100.0
$\alpha = 1$					
$E = 0.0842$					
($\kappa^2 = 0.0824$)					
B.S.	0.984	0.856	0.309	0.137–01	0.21–03
N.R.	0.985	0.864	0.323	0.136–01	0.17–03
P.F.	0.984	0.853	0.299	0.97–02	0.34–04
$\alpha = 3$					
$E = 0.432$					
($\kappa^2 = 0.385$)					
B.S.	0.996	0.962	0.705	0.139	0.50–02
N.R.	0.997	0.968	0.740	0.146	0.34–02
P.F.	0.995	0.949	0.621	0.56–01	0.23–03
$\alpha = 2\pi$					
$E = 2.0$					
($\kappa^2 = 1.0$)					
B.S.	0.998	0.983	0.848	0.339	0.285–01
N.R.	0.999	0.988	0.886	0.378	0.189–01
$E = 1.90$					
P.F.	0.614	0.398–01	0.11–03	0.13–06	0.13–09
$E = 1.95$					
P.F.	0.187	0.14–02	0.19–05	0.20–08	0.20–11

The comparison with the point-form results evidences a discrepancy that increases with the momentum transfer as well as with the coupling strength. It becomes especially large when approaching the extreme case where $M = 0$. Two features are worthwhile to be mentioned, that stem from examining the analytic expressions of the form factors, eq. (15). First, there is a contribution to the squared-charge radius which varies like $1/M^2$, as it was found numerically in ref. [13]. Second, the form factor drops more quickly with Q^2 than in the "exact" or in the "non-relativistic" calculations, roughly like $1/Q^6$ instead of $1/Q^4$, as expected from the examination of the Born amplitude. The discrepancy can be seen in table 1 for $E = 1.90\,m$ and $E = 1.95\,m$, where the effect becomes especially sizeable when approaching $Q^2 = 4\,M^2$ and beyond. Some departures from the reference calculation could be expected in the ultra-relativistic limit, due to the approximate character of the wave functions we used for instance. Their magnitude is typically given by the difference of the "non-relativistic" and "exact" calculations (B.S.), which can be obtained by examining the

Table 2. Inelastic form factors, $F_1(q^2)$ and $F_2(q^2)$, for a $l = 0 \to l = 0^*$ transition, and $\alpha = 3, E_i = 0.4322\,m$, $\kappa_i^2 = 0.385\,m^2$, $E_f = 0.1036\,m$, $\kappa_f^2 = 0.101\,m^2$ for the exact calculation and $0.096\,m^2$ for the "non-relativistic" one.

Q^2/m^2	0.01	0.1	1.0	10.0	100.0
B.S.					
F_1	0.032–01	0.298–01	0.145–00	0.584–01	0.21–02
F_2	0.369–00	0.340–00	0.165–00	0.665–02	0.22–04
N.R.					
F_1	0.032–01	0.296–01	0.151–00	0.550–01	0.12–02
F_2	0.369–00	0.342–00	0.174–00	0.636–02	0.14–04
P.F.					
F_1	0.101–01	0.372–01	0.140–00	0.283–01	0.14–03
F_2	0.324–00	0.293–00	0.119–00	0.22–03	−0.12–04

table. In comparison, the discrepancy for the point-form results is orders of magnitude larger however.

Results for an inelastic transition are given in table 2. The results obtained in the "non-relativistic" approach compare well with the "exact" ones over the full range of Q^2. The slight discrepancies for $Q^2 \leq 10\,m^2$ are quite similar to those observed for the elastic form factors. While the point-form results compare well with the other ones in the intermediate range, $0.1 < Q^2/m^2 < 3$, they fail at low and at large Q^2. In the former case, $F_1(q^2)$ does not go to zero when $Q^2 \to 0$, violating the current conservation, eq. (2). In the latter case, $F_2(q^2)$ evidences a change in sign around $Q^2 \simeq 10\,m^2$, also preventing one from fulfilling this relation. This can be traced back to the different behaviour of the intermediate form factors $\tilde{F}_1$ and $\tilde{F}_2$, which keep the same sign but scale like Q^{-6} and Q^{-8} at high Q^2, respectively. These results for an inelastic transition complement those for the elastic case. Again, a large discrepancy with "exact" results appears, but the bad behaviour of form factors at low as well as large momentum transfers is more clearly correlated with a violation of current conservation. Implementing this conservation law could remove some of the problems. However, it does not guarantee that the high-Q^2 behaviour would be correctly accounted for. In this domain, the constraints imposed by the consideration of the Born amplitude should also be considered.

4 Discussion and conclusion

The present study was motivated by the necessity to check the reliability of retaining the single-particle current for the calculation of form factors in the point form of relativistic quantum mechanics, which was recently employed in different works [12,13]. With this aim, we considered a simple model where the "exact" result is known. It is found that, for momentum transfers up to 3-4 times the constituent mass, the point-form approach does correctly for small bindings and small couplings. However, large discrepancies with the "exact" results appear as soon as the momentum transfer or the coupling increases. They can reach orders of magnitude. Detailed examination evidences three features:

- The form factors decrease more rapidly than they should. This points to an extra $1/Q^2$ dependence that is absent in the "exact" calculation.
- For strong couplings, corresponding to a sizeable presence of high momentum components in the wave function, the charge radius turns out to be much larger than the "exact" one.
- Finally, as emphasized by the results for an inelastic transition, current conservation is strongly violated.

While results presented here offer the advantage to be analytical, providing insight into the above features, we also looked at a different effective interaction. It was found that the above features qualitatively persist, indicating that they are not specific to the effective interaction employed in the calculation.

As expected, the "non-relativistic" approach is also found to do well in situations amenable to a non-relativistic calculation (small binding, low momentum transfers). When looking at typical relativistic limits ($M \to 0$ or $Q^2 \to \infty$), departures show up but never go beyond a factor two for the Q^2 range considered here (a result that by itself deserves an explanation). Surprisingly, for cases that have a typical relativistic character, it does considerably better than the point-form approach which, *a priori*, was expected to provide results closer to the "exact" ones.

Although it is not quite certain, there is good reason to believe that the "non-relativistic" calculation does relatively well at low Q^2 because it fulfils current conservation. With this respect, the failure of the point-form approach as applied here, following recent works published in the literature [12,13], has to reside in the incomplete character of the one-body current operator. Current conservation may be enforced by the replacement

$$\langle J^\mu \rangle \;\; \to \;\; \langle J^\mu \rangle - q^\mu \, \langle J_\nu \rangle \cdot \, q^\nu / q^2 . \qquad (17)$$

Apart from the unsatisfactory character of this recipe, due to the presence of a pole at $q^2 = 0$, it does not solve the problem of the too fast drop-off of the elastic charge form factor at high Q^2. Whether a minimal set of two-body currents will be sufficient to provide results in better agreement with the "exact" ones at low Q^2 is not clear, however. These currents should also correct for the failure of the point-form results to reproduce the Born amplitude expected from the underlying field theory model, which mainly concerns the region of high Q^2.

Arguments supporting the relevance of the comparison made here have been given in the text, especially in the introduction and in the section concerning the choice of the interaction model for the relativistic quantum-mechanical approach. The possibility remains that the above failure originates from missing something in the derivation of this interaction. If this was the case, the comparison of the "exact" results with those obtained in instant- or front-form approaches should also evidence similar features. Results obtained recently with these two approaches show, on the

contrary, reasonable agreement [19]. For the "zero-range" interaction model presented in ref. [20], the front-form results are identical to the exact ones while the instant-form ones in the Breit frame deviate by 10–15% at most. As the sensitivity to the interaction is essentially absent in this case (the wave function is uniquely determined by the mass of the system), this uncertainty is an indication of the sensitivity to the precise choice of the current when $q^\mu \neq 0$. For a long-range interaction, like the one used in this work, deviations from the "exact" results can reach a factor 2 for the largest momentum transfers considered here, both in the instant- and front-form approaches. This factor, which has nothing in common with the orders of magnitude discrepancy obtained in the point-form approach, most probably represents the sensitivity to the interaction model (and perhaps some two-body currents). The simplest choice has been made, without optimization to reproduce the "exact" spectrum. Thus, these results tend to validate the comparison of the Bethe-Salpeter results to those obtained in relativistic quantum-mechanics approaches together with the field theory motivation underlying the choice of the interaction in these cases.

While one expects that some of the remaining discrepancies should be accounted for by contributions due to two-body currents, it is striking that their role is considerably larger for the point-form than for the other approaches. This points to the way this approach, originally supposed to be described on an hyperboloid $x \cdot x = ct$, has been implemented in practice [11–13]. The procedure relies on employing a unique wave function, issued from an instant-form calculation, kinematically boosted to describe the initial and final states with the appropriate momenta. The results suggest that the procedure is unable or too crude to provide a single-particle current contribution with the right size, contrary to the other approaches [20].

The above results cannot be applied directly to the calculation of the deuteron or nucleon form factors [12,13]. However, the qualitative similarity in the results strongly suggests that the kinematical boost, which provided a nice description of the nucleon form factors, represents an incomplete account of relativistic effects. In particular, the kinematical boost does not account for boost effects related to the interaction part, as simply illustrated by the example of the total momentum itself which, as is well known, contains the interaction in the point-form approach [21]. As can be easily checked for a two-body system, the momentum acquired in a boost receives a contribution from the kinematical part, which represents the total momentum enhanced by a factor $2\,e_k/M$ ($2\,e_k$ is the sum of the free-particle energies in the c.m.), and an interaction part which cancels this enhancement so that the total momentum is recovered. It is not a surprise therefore if instant-form or front-form calculations [19], where both effects are accounted for simultaneously, do not evidence the peculiar features of the point-form approach ones when $M \to 0$ or $Q^2 \to \infty$. With respect to these two features, they are roughly in agreement with an "exact" calculation. In view of the previous discussion, the agreement for the nucleon form factors mentioned above thus appears, most likely, as accidental and results from neglecting significant contributions to the current. This conclusion is to be preferred with two respects:

- It leaves some room for the well-known contribution of the coupling of the nucleon to the photon through vector-meson exchange (vector-meson dominance mechanism), which roughly provides half of the proton's squared charge radius. In a constituent quark model, this effect can be partly accounted for by introducing quark form factors. In a more refined model, this supposes to include a quark-antiquark component in the nucleon wave function as explored in ref. [22].
- On the other hand, it leaves room for another relativistic effect related to the nature of the coupling of the constituents to the exchanged boson. This effect increases the form factor at high Q^2 rather than the opposite as in ref. [13]. It is known to explain the different asymptotic form factors in a non-relativistic and a relativistic calculation in QCD, which scale like $1/Q^8$ and $1/Q^4$, respectively (see for instance refs. [23,24]).

Altogether, two-body currents should produce quite sizeable contributions in present applications of the point-form approach. Their role seems to be more essential here than in other approaches. An open question is whether a different way of implementing the point-form approach [20] could minimize this role.

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